View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

ELSEVIER

Linear Algebra and its Applications 295 (1999) 7–30

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Thiele-type and Lagrange-type generalized inverse rational interpolation for rectangular complex matrices ☆

Chuanqing Gu

Department of Mathematics, Shanghai University, Box 30, 149, Yan Chang Road, Shanghai 200072, People's Republic of China

Received 15 April 1998; received in revised form 31 January 1999

Submitted by R.A. Brualdi

Abstract

A variety of matrix rational interpolation problems include the partial realization problem for matrix power series and the minimal rational interpolation problem for general matrix functions. Different from the previous work, in this paper we consider a new method of matrix rational interpolation, with rectangular real or complex interpolated matrices and distinct real or complex interpolation points. Based on an axiomatic definition for the generalized inverse matrix rational interpolants (GMRI), GMRI are constructed in the following two forms: (i) Thiele-type continued fraction expression; (ii) an explicit determinantal formula for the denominator scalar polynomials and for the numerator matrix polynomials, which are of Lagrange-type expression. As a direct application of GMRI, a matrix rational extrapolation is introduced. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

The matrix rational interpolation problems include the partial realization problem for matrix power series and Newton-Pade, Hermite-Pade, simultaneous Pade, M-Pade and multipoint Pade approximation problems with their

☆ The work is supported by the National Natural Science Foundation of China (19871054).

0024-3795/99/$ – see front matter © 1999 Elsevier Science Inc. All rights reserved.
PII: S 0 0 2 4 - 3 7 9 5 (9 9) 0 0 0 3 6 - 1

matrix generalizations [5,6]. The previous work studied the matrix rational interpolation problems with the same interpolation points. By means of the reachability and the observability indices of defined pairs of matrices, Antoulas et al. [1] have solved the minimal matrix rational interpolation problem. Using Loewner matrix, Anderson and Antoulas [4] considered the problem of passing from interpolation data for a real rational transfer-function matrix to a minimal state-variable realization of the transfer-function matrix. One of the matrix rational interpolation problems [1, p. 523] and [4] is as follows.

Given the quantities, with finite entries,

$$x_i \in k, \quad Y_i \in k^{s\times t} \tag{1.1}$$

with $x_i \neq x_j,\ i \neq j$, find all rational $s \times t$ matrices $Y(x)$ such that

$$Y(x_i) = Y_i, \quad i \in N. \tag{1.2}$$

In this paper, we consider a new method of the matrix rational interpolation problem, as in (1.1) and (1.2), with rectangular real or complex interpolated matrices and distinct real or complex interpolation points, which is an extension and improvement of generalized inverse vector rational interpolation discussed by Graves-Morris [17] and Graves-Morris and Jenkins [18]. Compared to previous methods, the method of generalized inverse matrix rational interpolants (GMRI) has the following advantages: first, it need not use multiplication of matrices in the construction process, thus, we do not have to define left and right interpolants; second, we have easy recursive algorithm for continued fractions and explicit determinantal formulas for finding GMRI; third, it can apply to singular matrices and is unique in some sense. On the other hand, the method of GMRI is of the divisibility constraint and degree constraint, which is due to the construction process of GMRI. The construction constraint shows that our method is not as effective as the method of minimal matrix rational interpolation [1] in some cases (see Example 5.3).

In Section 2, we define the GMRI and establish the uniqueness of GMRI. In Section 3, by means of a generalized reciprocal quotient for a matrix, which is found to be effective in continued fraction interpolation [8–13], we build the continued fraction expression of GMRI. In Section 4, we construct explicit determinantal formulas for denominator polynomials and for numerator matrix polynomials, which are of Lagrange-type expression. In the sequel, some examples are given to illustrate the results in this paper and compared to the method of minimal matrix rational interpolation method [1]. In the end, as a direct application of GMRI, we propose a matrix rational extrapolation which is introduced by Wuytack in the case of scalar quantities [20].

We establish some basic principles for the GMRI same as [8–13] as follows:

(i) If, for some fixed k, $k = 1, 2, \ldots, s$ or t, the kth row (or column) vector of the matrix $A(x_i)$ is the only non-zero row (or column) vector, then the matrix valued interpolant reduces to the corresponding vector valued rational interpolant [17,18].

(ii) The value of the matrix rational interpolant does not depend on the order in which the interpolation points are used to construct the interpolant.

(iii) There is some sense in which a specified rational interpolant is unique.

(iv) The poles of the $s \times t$ elements of the matrix valued interpolant normally occur at common positions in the x-plane.

2. Generalized inverse matrix rational interpolation problem

Given a data set, as in (1.1) and (1.2),

$$\{(z_i, A_i) : i = 0, 1, \ldots, n\}, \tag{2.1}$$

where interpolation points $z_i \in C$ with $z_i \neq z_j,\ i \neq j$, corresponding interpolated constant matrices $A_i = A(z_i) \in C^{s\times t}$.

Definition 2.1. The GMRI of type $[n/2k]$ is a matrix of rational function $R(z) = P(z)/q(z)$, where $P(z) = (p^{(uv)}(z)) \in C^{s\times t}$ is a complex matrix polynomial and $q(z)$ is a complex scalar polynomial, satisfying the following conditions:

$$\text{(i) } R(z_i) = A_i, \quad i = 0, 1, \ldots, n, \quad q(z_i) \neq 0, \tag{2.2}$$

$$\text{(ii) } \partial\{P\} = \max \partial\{p^{(uv)}\} \leqslant n, \quad \partial\{q\} = 2k, \tag{2.3}$$

$$\text{(iii) } q(z) \Big| \|P(z)\|^2, \tag{2.4}$$

$$\text{(iv) } q(z) = q^*(z), \tag{2.5}$$

where a superscript $*$ denotes complex conjugate,

$$\|P(z)\|^2 = (P(z) \mid P(z)) = \mathrm{tr}(P(z)^H P(z)) = \sum_{u=1}^{s} \sum_{v=1}^{t} \left| p^{(uv)}(z) \right|^2. \tag{2.6}$$

With regard to the divisibility condition (iii) in Definition 2.1, it is necessary that the scalar denominator of the interpolant divides the square of the norm of the numerator. The divisibility condition is due to the construction process of GMRI; however, the divisibility condition holds in the case of matrix interpolants, vector interpolants and scalar interpolants. In fact, (i) if $s \in R$ is a real number, it holds $ss = |s|^2,\ 1/s = s^{-1} = s/|s|^2$; (ii) if $b \in C$ is a complex number, it holds $bb^* = |b|^2,\ 1/b = b^{-1} = b^*/|b|^2$; (iii) if $\vec{v} \in C^d$ is a vector, it holds $\vec{v}\cdot\vec{v}^* = |\vec{v}|^2$, $1/\vec{v} = \vec{v}^{-1} = \vec{v}^*/|\vec{v}|^2$. (iv) Let $A = (a_{ij}), B = (b_{ij}) \in C^{s\times t}$ be two matrices, we define scalar product of matrices by means of dot product of vector as follows: $A \circ B = (a_{ij}b_{ij}) \in C^{s\times t}$, then remain to hold $A \circ A^* = \|A\|^2$, as in (2.6), and $1/A =: A^*/\|A\|^2$, as in (3.1). In a word, the divisibility constraint of GMRI in Definition 2.1 is based on the facts (i)–(iii) and their extension (iv).

Example 2.2. Find the [2/2] type GMRI $R(z) = P(z)/q(z)$, respectively, for the data

$$\text{(I) } A_0 = \begin{bmatrix} 2 & 0 \\ 0 & -i \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 1 & i \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix},$$

$$\text{(II) } \vec{v}_0 = (2,0), \quad \vec{v}_1 = (1,0), \quad \vec{v}_2 = (0,i),$$

$$\text{(III) } f_0 = 1, \quad f_1 = 2, \quad f_2 = -1$$

at points $x_0 = -1, \quad x_1 = 0, \quad x_2 = 1.$

Solution (1) (Solution (2) see Example 3.6):

(I) The cardinal functions are (see (4.1))

$$l_0(x) = (1/2)(x^2 - x), \quad l_1(x) = 1 - x^2, \quad l_2(x) = (1/2)(x^2 + x). \tag{2.7}$$

By (4.4) and (4.26) or (4.5), we get respectively

$$q(x) = \frac{3}{4}\begin{vmatrix} 0 & 24 & -7 \\ -2 & 0 & -1 \\ l_0(x) & l_1(x) & l_2(x) \end{vmatrix} = \frac{3}{2}(11x^2 + 6x + 7), \tag{2.8}$$

$$P(x) = \frac{3}{4}\begin{vmatrix} 0 & 24 & -7 \\ -2 & 0 & -1 \\ A_0 l_0(x) & A_1 l_1(x) & A_2 l_2(x) \end{vmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 5x^2 - 12x + 7 & 12x(x+1)i \\ 5x^2 + 12x + 7 & (-13x^2 + 6x + 7)i \end{bmatrix}. \tag{2.9}$$

$R_{\mathrm{M}}(x) = P(x)/q(x)$ satisfies:

(i) $R_{\mathrm{M}}(x_i) = A_i, \ i = 0,1,2;$

(ii) $\partial\{P\} = 2, \partial\{q\} = 2;$

(iii) $\|P\|^2 = 3(121x^4 + 44x^3 + 142x^2 + 28x + 49) = 3qg_{\mathrm{M}}, g_{\mathrm{M}} = 11x^2 - 2x + 7.$

(II) By (4.4) and (4.26), we get respectively

$$q(x) = \frac{1}{4}\begin{vmatrix} 0 & 4 & -5 \\ -1 & 0 & 2 \\ l_0(x) & l_1(x) & l_2(x) \end{vmatrix} = \frac{1}{4}(x^2 - 2x + 5),$$

$$\vec{P}(x) = \frac{1}{4}\begin{vmatrix} 0 & 4 & -5 \\ -1 & 0 & 2 \\ \vec{v}_0 l_0(x) & \vec{v}_1 l_1(x) & \vec{v}_2 l_2(x) \end{vmatrix} = \frac{1}{4}(3x^2 - 8x + 5, 2x(x+1)i).$$

$\vec{R}_{\mathrm{V}}(x) = \vec{P}(x)/q(x)$ satisfies:
(i) $\vec{R}_{\mathrm{V}}(x_i) = \vec{v}_i,\ i = 0, 1, 2$;
(ii) $\partial\{\vec{P}\} = 2, \partial\{q\} = 2$;
(iii) $\|\vec{P}\|^2 = 13x^4 - 40x^3 + 98x^2 - 80x + 25) = qg_{\mathrm{V}}, g_{\mathrm{V}} = 13x^2 - 14x + 5$.
(III) By (4.4) and (4.26), we get respectively

$$q(x) = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 9 \\ l_0(x) & l_1(x) & l_2(x) \end{vmatrix} = 4x^2 - 4x + 1,$$

$$P(x) = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 9 \\ f_0 l_0(x) & f_1 l_1(x) & f_2 l_2(x) \end{vmatrix} = 2x^2 - 5x + 2,$$

$R_{\mathrm{S}}(x) = P(x)/q(x)$ satisfies:
(i) $R_{\mathrm{S}}(x_i) = f_i,\ i = 0, 1, 2$;
(ii) $\partial\{P\} = 2, \partial\{q\} = 2$;
(iii) $\|P\|^2 = 4x^4 - 20x^3 + 33x^2 - 20x + 4 = qg_{\mathrm{S}}, g_{\mathrm{S}} = x^2 - 4x + 4$.

We pay particular attention to the fact that $R_{\mathrm{S}}(x) = P(x)/q(x)$ is a scalar rational interpolant with real coefficients. It satisfies the divisibility condition (iii) in Definition 2.1.

Theorem 2.3. *If a GMRI $R(z)$ of type $[n/2k]$ exists for data* (2.1), *then the rational function $R(z)$ is unique.*

Proof. The method is an extension of that of Graves-Morris and Jenkins [18], from the vector case to the matrix case.

Let $R(z) = P(z)/q(z)$ as in (2.2)–(2.5). By (2.5), we may express

$$q(z) = q_b(z)\prod_{i=0}^{n}(z - z_i)^{\beta_i}(z - z_i^*)^{\alpha_i}, \tag{2.10}$$

where $q_b(z)$ is a polynomial, $\alpha_i = 0$ if z_i is real but otherwise $\alpha_i = \beta_i$ (obviously $\beta_i \geqslant \alpha_i$),

$$q_b(z_i) \neq 0, \quad q_b(z_i^*) \neq 0, \quad i = 0, 1, \ldots, n. \tag{2.11}$$

Using (2.2), we may also define a matrix polynomial $P_b(z)$ by

$$P(z) = P_b(z) \prod_{i=0}^{n} (z - z_i)^{\beta_i}. \tag{2.12}$$

From (2.10), (2.12) and (2.3), (2.4), we get

$$\partial\{P_b\} \leqslant n - \beta, \quad \partial\{q_b\} = 2k - \beta - \alpha, \tag{2.13}$$

where

$$\beta = \sum_{i=0}^{n} \beta_i, \quad \alpha = \sum_{i=0}^{n} \alpha_i. \tag{2.14}$$

Let $\tilde{R}(z) = \tilde{P}(z)/\tilde{q}(z)$ be another GMRI for the data (2.1). By means of the equivalent of (2.2)–(2.5), (2.10)–(2.14), we derive

$$R(z) - \tilde{R}(z) = T(z) \left[q_b(z)\tilde{q}_b(z) \prod_{i=0}^{n} (z - z_i^*)^{\alpha_i + \tilde{\alpha}_i} \right]^{-1}, \tag{2.15}$$

where $T(z),\ M(z)$ are matrix polynomials defined by

$$T(z) = P_b(z)\tilde{q}_b(z) \prod_{i=0}^{n} (z - z_i^*)^{\tilde{\alpha}_i} - \tilde{P}_b(z) q_b(z) \prod_{i=0}^{n} (z - z_i^*)^{\alpha_i} \tag{2.16}$$

$$= M(z) \prod_{i=0}^{n} (z - z_i) \tag{2.17}$$

with

$$\partial\{M\} \leqslant 2k - \beta - \tilde{\beta} - 1. \tag{2.18}$$

The divisibility hypothesis (2.4) of $R(z) = P(z)/q(z)$ and $\tilde{R}(z) = \tilde{P}(z)/\tilde{q}(z)$ implies that

$$q(z)\tilde{q}(z) \Bigg| \left\| P(z)\tilde{q}(z) - \tilde{P}(z)q(z) \right\|^2. \tag{2.19}$$

Using (2.10)–(2.12) and (2.15)–(2.17) in (2.19), we obtain that

$$q_b(z)\tilde{q}_b(z)\Big|\|M(z)\|^2\prod_{i=0}^{n}(z-z_i)(z-z_i^*)^{1+\beta_i+\tilde{\beta}_i-\alpha_i-\tilde{\alpha}_i}. \tag{2.20}$$

If $M(z)\neq 0$, then we find from (2.11) and (2.20) that

$$q_b(z)\tilde{q}_b(z)\Big|\|M(z)\|^2, \tag{2.21}$$

from (2.13), (2.18) and (2.21), we get

$$2(2k-\beta-\tilde{\beta}-1)\geqslant(2k-\beta-\alpha)+(2k-\tilde{\beta}-\tilde{\alpha})$$

or

$$0\leqslant\alpha+\tilde{\alpha}-\beta-\tilde{\beta}-2$$

which is impossible because of

$$\beta_i\geqslant\alpha_i,\quad \tilde{\beta}_i\geqslant\tilde{\alpha}_i.$$

Hence $M(z)\equiv 0$ and $R(z)$ is unique. □

3. Thiele-type continuous fraction expression

In [8,9], we introduced a generalized reciprocal quotient for a matrix, as in (3.1), which was found to be effective in the matrix continued fraction interpolation [8–13]. In fact, the divisibility constraint of GMRI in Definition 2.1 is from the following construction:

$$1/A =: A^*/\|A\|^2,\quad A=(a_{ij})\in C^{s\times t},\quad A\neq 0, \tag{3.1}$$

where $\|A\|$ is as in (2.6) and A^* is the conjugate matrix of A.

Suppose the interpolation point set $\Phi=\{z_i=x_i,\ i=0,1,\ldots,n : x_i\in R\}$ in (2.1) in this section. By means of (3.1), we can recursively define the nth convergence of Thiele-type continued fractions:

$$R_n^{(0)}(x)=B_0(x_0)+\frac{x-x_0}{B_1(x_0x_1)}+\cdots+\frac{x-x_{n-1}}{B_n(x_0x_1,\ldots,x_n)} \tag{3.2}$$

with

$$\begin{aligned} &B_0(x_i)=A(x_i),\quad i=0,1,\ldots,n,\\ &B_1(x_0x_1)=(x_1-x_0)/(B_0(x_1)-B_0(x_0)),\\ &B_l(x_0x_1,\ldots,x_l)=(x_l-x_{l-1})/(B_{l-1}(x_0,\ldots,x_{l-2}x_l)\\ &\qquad\qquad -B_{l-1}(x_0,\ldots,x_{l-1})),\quad l\geqslant 2. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let* $B_l(x_0, \dots, x_l)$, $0 \leqslant l \leqslant n$, *exist and be different from zero* (*except for* $B_0(x_0)$) *and*

$$\begin{aligned} R_n^{(i)}(x) &= B_i(x_0, \dots, x_i) + \frac{x - x_i}{R_n^{(i+1)}(x)} \\ &= B_i(x_0, \dots, x_i) + \frac{x - x_i}{B_{i+1}(x_0, \dots, x_{i+1})} + \cdots + \frac{x - x_{n-1}}{B_n(x_0, \dots, x_n)} \end{aligned} \tag{3.4}$$

satisfy

$$R_n^{(i+1)}(x_i) \neq 0, \quad i = 0, 1, \dots, n-1. \tag{3.5}$$

Then $R_n^{(0)}(x)$ *as in* (3.2) *exists such that*

$$R_n^{(0)}(x_i) = A_i, \quad x_i \in \Phi.$$

Proof. Let the conditions hold. Thus (3.2) exists and becomes

$$\begin{aligned} R_n^{(0)}(x_i) &= B_0(x_0) + \frac{x_i - x_0}{B_1(x_0 x_1)} + \cdots + \frac{x_i - x_{i-1}}{R_n^{(i+1)}(x_i)} \\ &= B_0(x_0) + \frac{x_i - x_0}{B_1(x_0 x_1)} + \cdots + \frac{x_i - x_{i-1}}{B_i(x_0 \cdots x_i)} = A_i, \quad x_i \in \Phi. \quad \square \end{aligned}$$

Theorem 3.2 [9, p. 76]**.** *Let* $B_l(x_0, \dots, x_l) \in C^{s \times t}, 0 \leqslant l \leqslant n$ *and* $x_i \in \Phi$. *Define* $R_n^{(0)}(x_i)$ *as in* (3.2) *by a tail-to-head rationalization using* (3.1). *Then a matrix polynomial* $P(x)$ *and a real scalar polynomial* $q(x)$ *exist such that*
(i) $R_n^{(0)}(x) = P(x)/q(x)$;
(ii) $q(x) \,\Big|\, \|P(x)\|^2$.

Theorem 3.3 [8]**.** *Let* $R_n^{(0)}(x) = P(x)/q(x)$ *as in* (3.2).
(i) *If n is even*, $R_n^{(0)}(x_i)$ *is of* $[n/n]$.
(ii) *If n is odd*, $R_n^{(0)}(x_i)$ *is of* $[n/n-1]$.

In terms of Theorem 3.1–3.3, we define $R_n^{(0)}(x) = P(x)/q(x)$, as in (3.2), as a GMRI of type $[n/2k]$ for the data (2.1). We can easily prove the following lemma by induction.

Lemma 3.4 [15]**.** *If* $v(x_i) \neq 0$, $i = 1, 2, \dots, n$, *then*

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(\frac{U(x)}{v(x)} \right) \bigg|_{x = x_i} = N^{(k)}(x_i)$$

is equivalent to

$$U^{(k)}(x_i) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}(v(x)N(x))\bigg|_{x=x_i},$$

where $U(x)$ is a matrix polynomial and $v(x)$ is a scalar polynomial.

Theorem 3.5. *Assume that $x_i \in \Phi \subset (a,b)$, $i = 0,1,\ldots,n$ and $R_n^{(0)}(x) = P(x)/q(x)$ as in* (3.2) *be a GMRI of type $[n/2k]$, $q(x) \neq 0$, $x \in (a,b)$. If the matrix function $A(x)$ is of order $n+1$ continuous derivatives in $x \in (a,b)$, then for any $x \in (a,b)$ holds*

$$A(x) - R_n^{(0)}(x) = \frac{w_n(x)}{(n+1)!q(x)}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}[q(x)A(x)]\bigg|_{x=\xi}, \quad x \in \xi \tag{3.6}$$

where

$$w_n(x) = \prod_{i=0}^{n}(x - x_i).$$

Proof. Suppose, respectively,

$$\begin{aligned} G(t) &= q(t)[A(t) - R_n^{(0)}(t)] - q(x)[A(x) - R_n^{(0)}(x)]w_n(t)/w_n(x), \\ F(t) &= [q(t)A(t) - P(t)]w_n(x) - [q(x)A(x) - P(x)]w_n(t). \end{aligned} \tag{3.7}$$

Note that $q(t) \neq 0, t \in (a,b)$ and by $G(x) = 0, G^{(k)}(x_i) = 0$. Applying Lemma 3.4 to (3.7) we have

$$F(x) = 0, \qquad F^{(k)}(x_i) = 0, \quad i = 1,2,\ldots,n. \tag{3.8}$$

In (3.8), the total number of interpolating points and the point x is equal to $n+2$. By using the Rolle theorem, it follows from (3.8) that there exists $\xi \in (a,b)$ such that

$$F^{(n+1)}(\xi) = 0.$$

According to Theorem 3.3, $\deg\{P\} \leqslant n$, thus holds

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}(P(t)) = 0. \tag{3.9}$$

Substituting (3.9) into (3.7), it is derived that

$$q(x)A(x) - P(x) = \frac{w_n(x)}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}[q(x)A(x)]\bigg|_{x=\xi}. \quad \square$$

Example 3.6. Find the [2/2] type GMRI $R(z) = P(z)/q(z)$, respectively, for the data same as Example 2.2 at points $x_0 = -1,\ x_1 = 0,\ x_2 = 1$.

Solution (2) (Solution (1) see Example 2.2):

(I) By (3.2), (3.3) and using (3.1), we get

$$R_2^{(0)}(x) = \begin{bmatrix} 2 & 0 \\ 0 & -i \end{bmatrix} + \cfrac{x+1}{\frac{1}{6}\begin{bmatrix} -1 & 0 \\ 1 & -2i \end{bmatrix} + \cfrac{x}{\frac{1}{11}\begin{bmatrix} -17 & 12i \\ 5 & -2i \end{bmatrix}}}$$

$$= \frac{1}{11x^2+6x+7}\begin{bmatrix} 5x^2-12x+7 & 12x(x+1)i \\ 5x^2+12x+7 & (-13x^2+6x+7)i \end{bmatrix}.$$

(II) By (3.2), (3.3) and using $1/\vec{v} = \vec{v}^{-1} = \vec{v}^*/|\vec{v}|^2$, we get

$$\vec{R}_2^{(0)}(x) = (2,0) + \cfrac{x+1}{(-1,0) + \cfrac{x}{(-1,2i)}}$$

$$= \frac{1}{x^2-2x+5}(3x^2-8x+5, 2x(x+1)i).$$

(III) By (3.2), (3.3) and using $1/s = s^{-1} = s/\mid s\mid^2$, we get

$$R_2^{(0)}(x) = 1 + \cfrac{x+1}{1 + \cfrac{x}{(-1/2)}} = 1 + \frac{x+1}{1-2x}$$

$$= 1 + \frac{(x+1)(1-2x)}{(1-2x)(1-2x)} = \frac{2x^2-5x+2}{4x^2-4x+1}.$$

We find that in (III) of Example 3.6, if

$$R_2^{(0)}(x) = 1 + \frac{x+1}{1-2x} = \frac{2-x}{1-2x} = \frac{p(x)}{q(x)},$$

then $p(x), q(x)$ do not satisfy the divisibility condition. The case is caused because it does not use reciprocal quotient operation $1/s = s^{-1} = s/|s|^2$.

4. Explicit determinantal formulas

As the data set (2.1), the ith cardinal polynomial of Lagrange-type is defined by

$$l_i(z) = \prod_{l\neq i, l=0}^{n} (z - z_l)/(z_i - z_l), \quad i = 0, 1, \ldots, n. \tag{4.1}$$

The set $I = \{z_0, z_1, \ldots, z_n\}$ is separated into two disjoint component sets

$$I_1 = \{z_0, z_1, \ldots, z_J\}, \quad I_2 = \{z_{J+1}, z_{J+2}, \ldots, z_n\}, \tag{4.2}$$

where the set I_1 consists of interpolation points whose conjugates are not in I, the set I_2 consists of real interpolation points and complex conjugate pairs. For example $I = \{i, 2i, 0, 1-i, 1+i\}$, then

$$I_1 = \{i, 2i\}, \quad I_2 = \{0, 1-i, 1+i\}.$$

Either I_1 or I_2 may be empty. Let

$$A_i = A(z_i) = \left(a_i^{(uv)}\right) \in C^{s\times t} \tag{4.3}$$

and for the sake of simplicity, let

$$\sum_d = \sum_{u=1}^{s} \sum_{v=1}^{t}.$$

Theorem 4.1. *Let $R(z) = P(z)/q(z)$ be a GMRI of type $[n/2k]$ for data* (2.1). *Then hold*

$$q(z) = \begin{vmatrix} L_{00} & L_{01} & \cdots & L_{0,2k-1} & L_{0,2k} \\ L_{10} & L_{11} & \cdots & L_{1,2k-1} & L_{1,2k} \\ \vdots & \vdots & & \vdots & \vdots \\ L_{2k-1,0} & L_{2k-1,1} & \cdots & L_{2k-1,2k-1} & L_{2k-1,2k} \\ l_0(z) & l_1(z) & \cdots & l_{2k-1}(z) & l_{2k}(z) \end{vmatrix}, \tag{4.4}$$

$$P(z) = \sum_{i=0}^{n} l_i(z) q_i A_i, \quad q_i = q(z_i), \tag{4.5}$$

where

$$L_{ll} = 0, \quad l = 0, 1, \ldots, 2k-1, \tag{4.6}$$

$$L_{lm} = \sum_{i=0}^{n} l_i^*(z_l) l_m(z_i^*) \sum_d a_i^{(uv)*}\left(a_l^{(uv)} - a_m^{(uv)}\right),$$
$$l = 0, 1, \ldots, J, \quad m = 0, 1, \ldots, 2k, \tag{4.7}$$

$$L_{lm} = \left(\frac{d}{dz} l_m(z_l)\right) \sum_d \tilde{a}_l^{(uv)*}\left(a_l^{(uv)} - a_m^{(uv)}\right) + \sum_{i=0}^{n} \left(\frac{d}{dz} l_i^*(z_l)\right) l_m(z_i^*) \sum_d a_i^{(uv)*}\left(a_m^{(uv)} - a_l^{(uv)}\right), \tag{4.8}$$

$$l = J+1, J+2, \ldots, n, \quad m = 0, 1, \ldots, 2k,$$

if z_l *is real*, $\tilde{a}_l^{(uv)} = a_l^{(uv)*}$, *otherwise, for each* $z_l \in I_2, \tilde{a}_l^{(uv)}$ *is the element of matrix* $A(z_l^*)$.

Proof. (i) $n = 2k$.

By the interpolatory property of (2.2), we express (4.5) as matrix form

$$P(z) = \sum_{i=0}^{n} l_i(z) q_i A_i = \left(p^{(uv)}(z)\right), \tag{4.9}$$

where

$$p^{(uv)}(z) = \sum_{i=0}^{n} l_i(z) q_i a_i^{(uv)}. \tag{4.10}$$

Let

$$q(z) = \sum_{i=0}^{n} l_i(z) q_i. \tag{4.11}$$

From the divisibility hypothesis (2.4), we may define a polynomial $Q(z)$ of degree n by

$$q(z)Q(z) = \|P(z)\|^2, \tag{4.12}$$

where

$$Q(z) = \sum_{i=0}^{n} l_i(z) Q_i, \quad Q_i = Q(z_i). \tag{4.13}$$

Using (2.6) and (4.10), we get

$$\|P(z)\|^2 = \sum_d \left|p^{(uv)}(z)\right|^2 = \sum_d p^{(uv)}(z)\left[p^{(uv)}(z)\right]^*, \tag{4.14}$$

substituting (4.13) and (4.14) into (4.12), we find

$$Q_i = \sum_d p^{(uv)}(z_i)\left[p^{(uv)}(z_i)\right]^* = \sum_d a_i^{(uv)}\left[p^{(uv)}(z_i)\right]^*. \tag{4.15}$$

Note that (2.5) and (4.13), we have

$$q_l^* = q(z_l^*) = \sum_{i=0}^{n} l_i(z_l^*) q_i, \quad l = 0, 1, \ldots, J, \tag{4.16}$$

$$Q_l^* = Q(z_l^*) = \sum_{i=0}^{n} l_i(z_l^*)Q_i, \quad l = 0, 1, \ldots, J. \tag{4.17}$$

Substituting (4.15) into (4.17) and taking its conjugate, we get

$$\begin{aligned}\sum_d a_l^{(uv)}\left[p^{(uv)}(z_l)\right]^* = Q_l = &\sum_{i=0}^{n} Q_i^* l_i^*(z_l) \\ = &\sum_{i=0}^{n} l_i^*(z_l)\sum_d a_i^{(uv)^*} p^{(uv)}(z_i^*), \\ &\quad l = 0, 1, \ldots, J.\end{aligned} \tag{4.18}$$

Now substituting (4.9) into (4.18) and using (4.16) and (4.17), we derive that

$$\begin{aligned}&\sum_d a_l^{(uv)} \sum_{i=0}^{n} l_i^*(z_l) a_i^{(uv)^*} \sum_{m=0}^{n} q_m l_m(z_i^*) \\ &\quad = \sum_d \sum_{i=0}^{n} l_i^*(z_l) a_i^{(uv)^*} \sum_{m=0}^{n} l_m(z_i^*) q_m a_m^{(uv)}, \quad l = 0, 1, \ldots, J.\end{aligned} \tag{4.19}$$

By means of (4.19), we obtain the following linear equations:

$$\sum_{m=0}^{n} L_{lm} q_m = 0, \quad l = 0, 1, \ldots, J, \tag{4.20}$$

the coefficient of q_m in (4.20) is L_{lm}, as given by (4.7) and (4.6).

Although (4.19) holds for all j, it turns out that (4.19) is null for $l = J+1, J+2, \ldots, n$, the same as in the case of a vector. We differentiate (4.12) with respect to z, giving

$$\begin{aligned}&Q'(z)q(z) + Q(z)q'(z) \\ &\quad = \sum_d \left\{ \left(p^{(uv)}(z)\right)' p^{(uv)}(z)^* + p^{(uv)}(z)\left(p^{(uv)}(z)^*\right)' \right\}.\end{aligned} \tag{4.21}$$

By putting $\{z = z_l, l = J+1,\ J+2, \ldots, n\}$, we get

$$\begin{aligned}Q_l' + Q_l q_l'/q_l = \sum_d \Big\{ &\left(p^{(uv)}(z_l)\right)' p^{(uv)}(z_l)^*/q_l \\ &+ p^{(uv)}(z_l)\left(p^{(uv)}(z_l)^*\right)'/q_l \Big\}.\end{aligned} \tag{4.22}$$

Substituting (4.15) into (4.22), we find

$$\begin{aligned}
&\sum_{d}\sum_{i=0}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}z}l_i^*(z_l)\right)a_i^{(uv)^*}\sum_{m=0}^{n}l_m(z_i^*)a_m^{(uv)}q_m \\
&\quad+\sum_{d}a_l^{(uv)}\tilde{a}_l^{(uv)^*}\sum_{m=0}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}z}l_m(z_l)\right)q_m \\
&\;=\sum_{d}\sum_{m=0}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}z}l_m(z_l)\right)q_m a_m^{(uv)}\tilde{a}_l^{(uv)^*} \\
&\quad+\sum_{d}a_l^{(uv)}\sum_{i=0}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)l_i^*(z_l)a_i^{(uv)^*}\sum_{m=0}^{n}q_m l_m(z_i^*), \\
&\quad l=J+1,J+2,\ldots,n.
\end{aligned}\tag{4.23}$$

By means of (4.23), we obtain the following linear equations:

$$\sum_{m=0}^{n}L_{lm}q_m=0,\quad l=J+1,J+2,\ldots,n-1\tag{4.24}$$

the coefficient of q_m in (4.24) is L_{lm}, as given by (4.8) and (4.6).

Eqs. (4.20), (4.24) and (4.10) form a system of $n+1$ non-homogeneous equations for $q_0,q_1,\ldots,q_n$, as shown by

$$\begin{bmatrix} L_{00} & L_{01} & \cdots & L_{0n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{J0} & L_{J1} & \cdots & L_{Jn} \\ L_{J+1,0} & L_{J+1,1} & \cdots & L_{J+1,n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{n-1,0} & L_{n-1,1} & \cdots & L_{n-1,n} \\ l_0(z) & l_1(z) & \cdots & l_n(z) \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ \vdots \\ \\ q_{n-1} \\ q_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \\ 0 \\ q(z) \end{bmatrix}.\tag{4.25}$$

Solving (4.25), we obtain that $q(z)$ is given by the determinantal formula (4.4).

(ii) $n<2k$.

We extend the interpolation set for data $\{(z_i,A_i):i=0,1,\ldots,n\}$ as $z_{n+1},z_{n+2},\ldots,z_N,N=2k$. Thus we define

$$h(z)=\prod_{l=n+1}^{N}(z-z_l),$$

$$B_i=A_ih(z_i),\quad i=0,1,\ldots,n,\quad B_i=0,\quad i=n+1,n+2,\ldots,N.$$

For data $\{(z_i,B_i):i=0,1,\ldots,N\}$ we construct $q(z)$, as given by (4.4) and

$$P(z)=P_N(z)/h(z),$$

where

$$P_N(z) = \sum_{i=0}^{N} l_i(z) q_i A_i.$$

(iii) $n > 2k$.

Let $N = 2k$ for data $\{(z_i, A_i) : i = 0, 1, \ldots, n\}$. We define a matrix polynomial $G(z)$ as

$$G(z_i) = A_i, \quad i = N+1, N+2, \ldots, n, \quad \partial\{G\} = n - N,$$

$$D_i = A_i - G(z_i), \quad i = 0, 1, \ldots, N.$$

For data $\{(z_i, D_i) : i = 0, 1, \ldots, N\}$ we construct $q(z)$, as given by (3.4) and

$$P(z) = P_N(z) + G(z)q(z),$$

where

$$P_N(z) = \sum_{i=0}^{N} l_i(z) q_i A_i. \quad \square$$

Theorem 4.2. *Let $R(z) = P(z)/q(z)$ be a GMRI of type $[n/2k]$ for data (2.1). Then hold*

$$P(z) = \begin{vmatrix} L_{00} & L_{01} & \cdots & L_{0,2k-1} & L_{0,2k} \\ L_{10} & L_{11} & \cdots & L_{1,2k-1} & L_{1,2k} \\ \vdots & \vdots & & \vdots & \vdots \\ L_{2k-1,0} & L_{2k-1,1} & \cdots & L_{2k-1,2k-1} & L_{2k-1,2k} \\ A_0 l_0(z) & A_1 l_1(z) & \cdots & A_{2k-1} l_{2k-1}(z) & A_{2k} l_{2k}(z) \end{vmatrix} \tag{4.26}$$

and satisfy

$$R(z_i) = P(z_i)/q(z_i) = A_i, \quad i = 0, 1, \ldots, n.$$

Proof. By means of (4.25) and (4.5), we get easily (4.26). From (4.26) or (4.5)

$$P(z) = \sum_{i=0}^{n} l_i(z) q_i A_i, \quad q_i = q(z_i),$$

it is held that

$$P(z_i) = q_i A_i = q(z_i) A(z_i), \quad i = 0, 1, \ldots, n. \quad \square$$

If all the points $x_0, x_1, \ldots, x_{2k}$ are real, the denominator polynomial for a $[2k/2k]$ GMRI takes the simple form.

Lemma 4.3. *Let $R(z) = P(z)/q(z)$ be a GMRI of type* $[n/2k]$ *for data* (2.1) *with all the interpolation points $x_0, x_1, \ldots, x_n$ are real and let $n=2k$. Then hold*

$$q(z) = \begin{vmatrix} 0 & l_1'(x_0)\|A_0 - A_1\|^2 & \cdots & l_n'(x_0)\|A_0 - A_n\|^2 \\ l_0'(x_1)\|A_1 - A_0\|^2 & 0 & \cdots & l_n'(x_1)\|A_1 - A_n\|^2 \\ \vdots & \vdots & \vdots & \vdots \\ l_0'(x_{n-1})\|A_{n-1} - A_0\|^2 & l_1'(x_{n-1})\|A_{n-1} - A_1\|^2 & \cdots & l_n'(x_{n-1})\|A_{n-1} - A_n\|^2 \\ l_0(x) & l_1(x) & \cdots & l_n(x) \end{vmatrix}, \tag{4.27}$$

$$P(z) = \begin{vmatrix} 0 & l_1'(x_0)\|A_0 - A_1\|^2 & \cdots & l_n'(x_0)\|A_0 - A_n\|^2 \\ l_0'(x_1)\|A_1 - A_0\|^2 & 0 & \cdots & l_n'(x_1)\|A_1 - A_n\|^2 \\ \vdots & \vdots & \vdots & \vdots \\ l_0'(x_{n-1})\|A_{n-1} - A_0\|^2 & l_1'(x_{n-1})\|A_{n-1} - A_1\|^2 & \cdots & l_n'(x_{n-1})\|A_{n-1} - A_n\|^2 \\ A_0 l_0(x) & A_1 l_1(x) & \cdots & A_n l_n(x) \end{vmatrix}, \tag{4.28}$$

where

$$\|A_l - A_m\|^2 = \sum_d \left|a_l^{(uv)} - a_m^{(uv)}\right|^2. \tag{4.29}$$

Theorem 4.4. *If* $q(z_i) \neq 0$ *for all interpolating points in* (2.1), *there exists a GMRI* $R(z) = P(z)/q(z)$ *of type* $[n/2k]$, *where* $P(z)$ *as in* (4.5) *or* (4.26) *and* $q(z)$ *as in* (4.4), *respectively*.

Proof. In fact, suppose $q(z_i) \neq 0$ for all interpolating points in (2.1) and substitute the interpolation points into (4.5) or (4.26) respectively, we obtain at once

$$P(z_i) = \sum_{k=0}^{n} l_k(z_i) q_k A_k = q(z_i) A(z_i), \quad i = 0, 1, \ldots, n. \tag{4.30}$$

Eq. (4.30) implies that

$$P(z_i)/q(z_i) = A(z_i), \quad i = 0, 1, \ldots, n. \quad \square$$

5. Examples and comparization

Example 5.1. Find the denominator $q(z)$ of the [2/2] type GMRI for data A_0, A_1, A_2 with $A_i = (a_i^{(uv)}),\ i = 0, 1, 2$ at points $z_0 = i,\ z_1 = 2i,\ z_2 = 0$.

Solution: By (4.1), we get the cardinal functions

$$\begin{aligned} l_0(z) &= z^2 - 2iz, \\ l_1(z) &= (-1/2)(z^2 - iz), \\ l_2(z) &= (-1/2)(z - i)(z - 2i). \end{aligned} \tag{5.1}$$

From (4.4), we form

$$q(z) = \begin{vmatrix} 0 & -3\|A_0 - A_1\|^2 & L_{02} \\ -24\|A_1 - A_0\|^2 & 0 & L_{12} \\ l_0(z) & l_1(z) & l_2(z) \end{vmatrix}, \tag{5.2}$$

where

$$L_{02} = 3\sum_d \left(a_2^{(uv)^*} - 3a_0^{(uv)} + 2a_1^{(uv)^*}\right)\left(a_0^{(uv)} - a_2^{(uv)}\right),$$

$$L_{12} = 6\sum_d \left(2a_2^{(uv)^*} - 4a_0^{(uv)} + 3a_1^{(uv)^*}\right)\left(a_1^{(uv)} - a_2^{(uv)}\right).$$

Let the above A_0, A_1, A_2 be, respectively,

$$\begin{aligned} A_0 &= \begin{bmatrix} 2 & 0 & 3-i \\ 0 & -i & -4-i \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 1 & 0 & 1-3i \\ 1 & i & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & i & l+i \\ 1 & 0 & -2i \end{bmatrix}. \end{aligned} \tag{5.3}$$

Find the [2/2] type GMRI $R(z) = P(z)/q(z)$ for the points $z_0 = i,\ z_1 = 2i$, $z_2 = 0$.

From (5.2), we get

$$q(z) = 72\begin{vmatrix} 0 & -31 & 32i-66 \\ -31 & 0 & 15i+3 \\ l_0(z) & l_1(z) & l_2(z) \end{vmatrix}$$

$$= -2232[(3+15i)l_0(z) + (-66+32i)l_1(z) + 31l_2(z)]$$

$$= -1116[(41-2i)z^2 + (28+15i)z + 62].$$

By (4.26) and (5.3), we get

$$P(z) = 72\begin{vmatrix} 0 & -31 & 32i-66 \\ -31 & 0 & 15i+3 \\ A_0 l_0(z) & A_1 l_1(z) & A_2 l_2(z) \end{vmatrix}$$

$$= -1116\begin{bmatrix} p^{(11)}(z) & p^{(12)}(z) & p^{(13)}(z) \\ p^{(21)}(z) & p^{(22)}(z) & p^{(23)}(z) \end{bmatrix},$$

where

$$p^{(11)}(z) = (78+28i)z^2 + (88-90i)z,$$

$$p^{(12)}(z) = -31i(z-i)(z-2i),$$

$$p^{(13)}(z) = -(13+177i)z^2 - (155-27i)z + 62(1+i),$$

$$p^{(21)}(z) = (35-32i)z^2 - (32-27i)z + 62,$$

$$p^{(22)}(z) = (62+60i)z^2 + (54-92i)z,$$

$$p^{(23)}(z) = (6-64i)z^2 - (66+12i)z - 124i.$$

We find that $R(z) = P(z)/q(z)$ satisfies: $R(z_i) = A_i,\ i = 0, 1, 2$ (see (5.3)).

Example 5.2. Find the [2/2] type GMRI $R(z) = P(z)/q(z)$ for the data

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{5.4}$$

at points $x_0 = -1,\ x_1 = 0,\ x_2 = 1.$

Solution (1): By (4.27), (4.28) and (2.7), we get respectively

$$q(x) = \begin{vmatrix} 0 & 6 & -1 \\ -3/2 & 0 & 3/2 \\ l_0(x) & l_1(x) & l_2(x) \end{vmatrix} = \frac{3}{2}(5x^2+1), \tag{5.5}$$

$$P(x) = \begin{vmatrix} 0 & 6 & -1 \\ -3/2 & 0 & 3/2 \\ A_0 l_0(x) & A_1 l_1(x) & A_2 l_2(x) \end{vmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 1-x^2 & 3x(x+1) \\ 2x^2+3x+1 & 1-x^2 \end{bmatrix}. \tag{5.6}$$

Solution (2): By (3.2), (3.3) and using (3.1), we get

$$R_2^{(0)}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{x+1}{\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \frac{x}{\frac{1}{5}\begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}}}$$

$$= \frac{1}{5x^2+1}\begin{bmatrix} 1-x^2 & 3x^2+3x \\ 2x^2+3x+1 & 1-x^2 \end{bmatrix}. \tag{5.7}$$

We find that (5.5), (5.6) and (5.7) satisfy, respectively, the data set (5.4).

Example 5.3 [6, Example 1]**.** The data (1.1) and (1.2) are as

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2, \quad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

According to the method of the minimal rational interpolation [1], the corresponding pair (F, G) and a column reduced matrix $\Theta(x)$ are, respectively.

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix},$$

$$\Theta(x) = \begin{bmatrix} -1 & 0 & -x^2+3x-2 & x^3-3x^2+2x \\ -1 & -x & x^2-3x & 0 \\ 1 & x & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

A parametrization of all minimal solutions is given by $Y^{\min}(x) = \Theta_1(x)\Theta_2^{-1}(x)$, where

$$\begin{bmatrix} \Theta_1(x) \\ \Theta_2(x) \end{bmatrix} = \begin{bmatrix} -1 & -x^2+3x-2 \\ -1 & x^2-3x-x(ax+b) \\ 1 & 2+x(ax+b) \\ 1 & 0 \end{bmatrix}.$$

The parameters a, b must satisfy

$$a+b \neq -2, \quad 2a+b \neq -1.$$

However, by (4.1), the cardinal functions are

$$l_0(x) = (1/2)(x^2-3x+2),$$
$$l_1(x) = -x^2+2x,$$
$$l_2(x) = (1/2)(x^2-x).$$

From (4.4), we derive that

$$q(x) = 4l_1(x) + 16l_2(x) = 4x^2.$$

As the existence condition (Theorem 4.4), there does not exist a GMRI for this example because of $q(x_0) = q(0) = 0$.

6. A matrix rational extrapolation

Assume that a convergent matrix sequence $\{A_n\} : A_n \to A (n \to \infty)$, $A_n, A \in C^{s\times t}$. We want to form a new matrix sequence $\{H_n\}$, derived from $\{A_n\}$, which has also A as limit and whose convergence is faster than of $\{A_n\}$. The determination of $\{H_n\}$ is to use rational extrapolation to the limit, which had been fulfilled by Wuytack [20] in the case of scalar quantities.

Let $\{x_n\} : x_n \to \infty \ (n \to \infty), \quad x_n \in R$ be a sequence of points. Define a matrix sequence of interpolating function $\{T_n\}$ such that

$$T_n(x_i) = A_i, \quad i = 0, 1, \ldots, n, \quad n = 0, 1, 2, \ldots$$

The new matrix sequence $\{H_n\}$ can be defined by

$$H_n = \operatorname*{Lim}_{x\to\infty} T_n(x), \quad n = 0, 1, 2, \ldots \tag{6.1}$$

if these limits exist and are finite.

In the following proof the method is different from scalar quantities [20], because we cannot use the three recurrence relations for continued fractions. By the results for Section 3, $T_n(x)$ in (6.1) is replaced by $R_n^{(0)}(x)$ in (3.2).

Theorem 6.1. *Suppose the conditions of Theorem* 3.1 *are satisfied. Then for even value of n, hold*

$$H_n = \operatorname*{Lim}_{x \to \infty} R_n^{(0)}(x) = B_0 + B_2 + B_4 + \cdots + B_n, \tag{6.2}$$

where $B_l = B_l(x_0, \ldots, x_l),\ 0 \leqslant l \leqslant n$, *as in* (3.3).

Proof. The proof is recursive. Assume n is even. The result is obviously for $n = 0$.

For $n = 2$, let $B_2 = N_0/D_0, 1/N_0 = N_0/\|N_0\|^2$.

We have

$$\begin{aligned} R_2^{(0)}(x) &= B_0 + \frac{x - x_0}{B_1} \, \frac{}{+} \, \frac{x - x_1}{B_2} \\ &= B_0 + \frac{(x - x_0)\|N_0\|^2}{B_1\|N_0\|^2 + (x - x_1)N_0 D_0} \\ &\quad \to B_0 + \frac{N_0}{D_0} = B_0 + B_2 \quad (x \to \infty). \end{aligned} \tag{6.3}$$

For $n = 2(k-1)$, suppose

$$R_{2(k-1)}^{(0)}(x) \ \to \ B_0 + B_2 + \cdots + B_{2(k-1)} \quad (x \to \infty). \tag{6.4}$$

By Theorem 3.3, $R_{2(k-1)}^{(0)}(x)$ is of type $[2(k-1)/2(k-1)]$.

For $n = 2k$, let

$$\frac{N_{2k-2}}{D_{2k-2}} = B_2 + \frac{x - x_2}{B_3} \, \frac{}{+ \cdots +} \, \frac{x - x_{2k-1}}{B_{2k}}.$$

By Theorem 3.3, N_{2k-2}/D_{2k-2} is of type $[2(k-1)/2(k-1)]$. Using the above inductive hypothesis (6.4), we find that

$$\frac{N_{2k-2}}{D_{2k-2}} \ \to \ B_2 + B_4 + \cdots + B_{2k-2} + B_{2k} \tag{6.5}$$

In terms of (6.3) and (6.5), we derive that

$$\begin{aligned} R_{2k}^{(0)}(x) &= B_0 + \frac{x - x_0}{B_1} \, \frac{}{+} \, \frac{x - x_1}{N_{2k-2}/D_{2k-2}} \\ &= B_0 + \frac{(x - x_0)\|N_{2k-2}\|^2}{B_1\|N_{2k-2}\|^2 + (x - x_1)N_{2k-2}D_{2k-2}} \\ &\quad \to B_0 + \frac{N_{2k-2}}{D_{2k-2}} = B_0 + B_2 + \cdots + B_{2k} \quad (x \to \infty). \quad \square \end{aligned}$$

Theorem 6.2. *Suppose the conditions of Theorem* 4.4 *are satisfied and* $R_n(x) = P(x)/q(x), z = x \in R$, *where* $P(x), q(x)$ *as in* (4.5) *or* (4.26) *and* (4.4), *respectively. Then for even value of* n, *hold*

$$H_n = \operatorname*{Lim}_{x \to \infty} R_n(x) = \frac{1}{L_n} \sum_{i=0}^{n} \frac{q(x_i)A_i}{\frac{\mathrm{d}}{\mathrm{d}x} w(x_i)}, \tag{6.6}$$

where L_n *is the algebraic complement for* x^n *in* $q(x)$, *as in* (4.4).

Proof. From (4.5) or (4.26)

$$P(x) = \sum_{i=0}^{n} l_i(x) q(x_i) A_i. \tag{6.7}$$

The coefficient of x^n in (6.7) is

$$\sum_{i=0}^{n} \frac{q(x_i)A_i}{\frac{\mathrm{d}}{\mathrm{d}x} w(x_i)}.$$

By the determinantal formula (4.4) for $q(x)$, we obtain (6.6). □

Example 6.3. Let the data set be as Example 2.2 (I). From (2.8), $L_2 = 33/2$ is the algebraic complement for x^2 in $q(x)$. By (6.6), we get from (2.8) and (2.9) that

$$H_2 = \operatorname*{Lim}_{x \to \infty} R_2(x) = \frac{1}{L_2} \sum_{i=0}^{2} \frac{q(x_i)A_i}{\frac{\mathrm{d}}{\mathrm{d}x} w(x_i)} = \frac{1}{11} \begin{bmatrix} 5 & 12i \\ 5 & -13i \end{bmatrix}.$$

By means of Theorem 6.1 and the generalized reciprocal quotient for a matrix (3.1), we construct two algorithms to form the matrix sequence $\{B_n\}$ in (6.2). Their formation rules are the same as the scalar case [20].

Algorithm 6.4. *Let* $T_0^0 = A_0,\ H_0 = A_0$,

$$T_0^k = A_k, \quad k = 1, 2, \ldots,$$
$$T_j^k = (x_k - x_{j-1}) / \left(T_{j-1}^k - T_{j-1}^{j-1}\right), \quad j = 1, 2, \ldots, k,$$
$$H_k = T_k^k + H_{k-2}, \quad k \text{ is even}.$$

Algorithm 6.5. *Let* $T_0^0 = A_0,\ H_0 = A_0$,

$$T_0^k = A_k, \quad k = 1, 2, \ldots,$$
$$T_1^k = (x_k - x_{k-1}) / (A_k - A_{k-1}), \quad k = 1, 2, \ldots,$$

Table 1
Illustration of Algorithm 6.5

k	T_0^k	T_1^k	T_2^k
0	$\frac{1}{2}\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$		
1	$\frac{1}{4}\begin{bmatrix}2&1&1\\1&2&1\\1&1&2\end{bmatrix}$	$\frac{2}{9}\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$	
2	$\frac{1}{8}\begin{bmatrix}2&3&3\\3&2&3\\3&3&2\end{bmatrix}$	$\frac{4}{9}\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}$	$\frac{1}{12}\begin{bmatrix}2&5&5\\5&2&5\\5&5&2\end{bmatrix}$

$$T_j^k = T_{j-2}^{k-1} + (x_k - x_{k-j})/\left(T_{j-1}^k - T_{j-1}^{j-1}\right), \quad j = 2, 3, \ldots, k,$$
$$H_k = T_k^k + H_{k-2}, \quad k \text{ is even}.$$

We remark that $B_k = T_k^k,\ k = 0, 1, \ldots$ in Algorithm 6.4 and

$$B_0 = T_0^0, \quad B_1 = T_1^1, \quad B_k = T_k^k - T_{k-2}^{k-2}, \quad k = 2, 3, \ldots$$

in Algorithm 6.5, thus, for the two algorithms hold

$$H_n = B_0 + B_2 + B_4 + \cdots + B_n, \text{if } n \text{ is even}.$$

Example 6.6. Find H_2 for a transition matrix of Markov process

$$T_0^n = \begin{bmatrix} t_n & t_{n+1} & t_{n+1} \\ t_{n+1} & t_n & t_{n+1} \\ t_{n+1} & t_{n+1} & t_n \end{bmatrix}, \quad t_n = \frac{1}{3}\left[1 + \frac{(-1)^{n+1}}{2^n}\right], \quad n = 0, 1, \ldots \tag{6.8}$$

Obviously

$$A = \lim_{n\to\infty} T_0^n = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}.$$

We take $x_n = n$ for $n = 0, 1, 2$ then the table for $T_i^k, i = 0, 1, 2$, constructed by means of Algorithm 6.5, is given in Table 1.

Acknowledgements

I am grateful to the referee for many valuable suggestions and for bringing to my attention Refs. [1,2]. I am also grateful to Prof. Graves-Morris for Refs. [17,18].

References

[1] A.C. Antoulas, J.A. Ball, J. Kang, J.C. Willems, On the solution of the minimal rational interpolation problem, Linear Algebra Appl. 137/138 (1990) 511–573.
[2] A.C. Antoulas, Rational interpolation and the Euclidean algorithm, Linear Algebra Appl. 108 (1988) 157–171.
[3] A.C. Antoulas, On recursiveness and related topics in linear systems, IEEE Trans. Automat. Control 31 (1986) 1121–1135.
[4] B.D.O. Anderson, A.C. Antoulas, Rational interpolation and state-variable realizationas, Linear Algebra Appl. 137/138 (1990) 479–509.
[5] B. Beckermann, G. Labahn, Recursiveness in matrix rational interpolation problems, J. Comput. Appl. Math. 77 (1997) 5–34.
[6] B. Beckermann, G.A. Labahn, Uniform approach for Hermite Pade and simultaneous Pade approximants and their matrix-type generalizations, Numer. Algorithms 3 (1992) 45–54.
[7] A. Bultheel, M. Van Barel, Pade techniques for model reduction in linear system theory: A survey, J. Comput. Appl. Math. 14 (1986) 401–438.
[8] Gu Chuanqing, Chen Zhibing, Matrix valued rational interpolants and its error formula, Math. Numer. Sinica 17 (1995) 73–77.
[9] Gu Chuanqing, Bivariate Thiele-type matrix valued rational interpolants, J. Comput. Appl. Math. 80 (1997) 71–82.
[10] Gu Chuanqing, On matrix valued osculatory rational interpolant, Numer. Math. J. Chinese Univ. 18 (1996) 135–141.
[11] Gu Chuanqing, Generalized inverse matrix valued Pade approximants, Numer. Sinica 19 (1997) 19–28.
[12] Gu Chuanqing, Multivariate generalized inverse matrix valued rational interpolation, Numer. Math. J. Chinese Univ. 19 (1997) 241–250.
[13] Gu Chuanqing, Zhu Gongqin, Matrix valued rational approximation, J. Math. Res. Exposition 16 (1996) 301–306.
[14] Gu Chuanqing, Multivariate generalized inverse vector-valued rational interpolants, J. Comput. Appl. Math. 84 (1997) 137–146.
[15] Gu Chuanqing, Matrix valued Salzer theorem, J. Hefei University of Technology 16 (1993) 176–179.
[16] Luca Dieci, Structure preserving piecewise polynomial interpolation for definite matrices, Linear Algebra Appl. 202 (1994) 25–32.
[17] P.R. Graves-Morris, Vector valued rational interpolants I, Numer. Math. 41 (1983) 331–334.
[18] P.R. Graves-Morris, C.D. Jenkins, Vector valued rational interpolants III, Constr. Approx. 2 (1986) 263–289.
[19] A. Messeoudi, Matrix recursive projection and interpolation algorithms, Linear Algebra Appl. 202 (1994) 71–89.
[20] L. Wuytack, A new technique for rational extrapolation to the limit, Numer. Math. 17 (1970) 215–221.